

Quantum Deformation of the Effective Theory on Non-Abelian string and 2D-4D correspondence

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Abstract

We explore non-Abelian strings in the $r = N - 1$ vacuum of $\mathcal{N} = 2$ supersymmetric QCD with the gauge group $U(N)$ and N_f flavors of quarks ($N_f \geq N$), where r is the number of condensed quarks. $\mathcal{N} = 2$ supersymmetry is broken down to $\mathcal{N} = 1$ by a small mass term for the adjoint matter. We discover that the low-energy two-dimensional theory on the string world-sheet receives nonperturbative corrections from the bulk, through the bulk gaugino condensate. This is in contradistinction with the $r = N$ vacuum situation, in which nonperturbative effects on the world sheet are determined by internal dynamics of the world-sheet theory. The 2D-4D correspondence (the coincidence of spectra of two-dimensional kinks and four-dimensional monopoles) remains valid in the BPS sector. Nonperturbative bulk effects deforming the weighted CP model on the world sheet are found by virtue of the method of resolvents suggested by Gaiotto, Gukov and Seiberg for surface defects [1]. In the $r = N$ vacuum the gaugino condensate in the bulk vanishes, and there are no “outside” nonperturbative corrections on the world sheet.

1 Introduction

Non-Abelian strings [2, 3, 4, 5] were first found in $\mathcal{N} = 2$ supersymmetric QCD, for reviews see e.g. [6, 7, 8, 9]. In the simplest version they appear in the theory with the $U(N)$ gauge group and $N_f = N$ quark flavors, with the Fayet-Iliopoulos parameter (FI) [10] $\xi \neq 0$. The non vanishing FI parameter triggers condensation of N flavors of (s)quarks, color-flavor locking occurs so that both the $U(N)$ gauge group and the flavor $SU(N)$ group are broken but the diagonal global subgroup $SU(N)_{C+F}$ survives.

The global $SU(N)_{C+F}$ symmetry unbroken in the vacuum but broken on the string is the reason why non-Abelian strings (i.e. those with orientational moduli) exist. The orientational zero modes on the string solution allow one to rotate its color flux inside the non-Abelian $SU(N)$ group with no change in energy. Dynamics of these orientational moduli fields is described by an effective two-dimensional $CP(N-1)$ model [2, 3, 4, 5]. The emergence of the $CP(N-1)$ model is easy to understand. The Z_N string solutions in the theory with the $U(N)$ gauge group break global $SU(N)_{C+F}$ down to $SU(N-1) \times U(1)$. This is why the orientational moduli live on

$$CP(N-1) = \frac{SU(N)_{C+F}}{SU(N-1) \times U(1)}. \quad (1.1)$$

The coset (1.1) is the target space of the sigma model on the world sheet of the simplest Non-Abelian string.

In the past decade many generalizations of the simplest model were worked out. We will continue along the lines of [11] to discover a conceptual novelty: nonperturbative effects from the bulk deform the target space of the world-sheet model (in special cases where such deformations are possible).

In order to explain when they are possible we need to remember the following:

The $CP(N-1)$ is robust in the sense that the Ricci tensor and all higher target space covariants are proportional to the Kähler metric. The world sheet dynamics is fully characterized by a single parameter, the coupling constant. In Ref. [11] in which the case $N_f > N$ and $r = N$ was considered (where r is the number of condensed quarks), we get the so-called weighted $CP(N, \tilde{N})$ model (WCP) with $\tilde{N} = N_f - N$. The corresponding target space is not robust even in perturbation theory: extra higher target space invariants (not reducible to the previous) appear in higher loops. Thus, this sigma model is not renormalizable in the conventional sense. The $r = N$ vacuum

considered in [11] implies complete Higgsing of the bulk theory. There exists a limit (all quark masses large and unequal) in which we could derive the world-sheet $WCP(N, \tilde{N})$ model that was weakly coupled.¹ Quantum corrections could be obtained “inside” this two-dimensional model *per se*. Then, in the BPS sector analytic continuation was possible to smaller masses (and mass differences).

In this paper we will consider the $r = N - 1$ vacuum, with the residual unbroken $U(1)$ in the bulk vacuum. In this case the situation with the world-sheet model turns out dramatically different. At the classical level it is still WCP, but a certain corner of its target space exhibits strong coupling for any values of the bulk parameters. It turns out that nonperturbative corrections in the bulk (represented by the gluino condensate) penetrate into the two-dimensional field theory deforming the standard representation for the WCP model. Correspondingly, the BPS spectrum receives nonperturbative corrections that go beyond those occurring in the conventional WCP.

Both theories – that of Ref. [11] and of the present work – share a remarkable feature. The two-dimensional sigma models we deal with are derived as world-sheet theories on the non-Abelian strings. We are mainly focused on the $\mathcal{N} = (2, 2)$ limit of these sigma models corresponding to the $\mu \rightarrow 0$ limit in the bulk theory, where μ is the deformation parameter (see below). The $\mathcal{N} = (2, 2)$ limit of the two-dimensional sigma models of interest exists and is well-defined. At the same time, in the limit of vanishing μ strings in the bulk disappear (since so do all (s)quark condensates). Thus, the situation we encounter with reminds the Cheshire cat’s smile. The smile is there while the cat is gone! The model we discuss here is even more spectacular since even at $\mu \neq 0$ one of the strings is absent.

As was already mentioned, in the simplest model with $N_f = N$ in the $r = N$ vacuum we obtain the $CP(N - 1)$ model on the world sheet of the non-Abelian string with $\mathcal{N} = (2, 2)$ supersymmetry. Due to the (s)quark condensation in the bulk theory, the monopoles are confined. In the $U(N)$ theories confined elementary monopoles are seen as junctions of two distinct non-Abelian strings, rather than string endpoints. They are also seen in the world-sheet theory – as kinks interpolating between two distinct vacua of the $CP(N - 1)$ model.²

¹ To be more exact, the theory on the semilocal non-Abelian string is the so-called zn model [12, 13]. It reduces to the $WCP(N, \tilde{N})$ model at $N \rightarrow \infty$ and, at finite N , in the BPS-protected sector.

² Note that the $\mathcal{N} = (2, 2)$ supersymmetric $CP(N - 1)$ model has N vacua associated

This picture leads to an absolute coincidence of the BPS spectra of the bulk $\mathcal{N} = 2$ QCD (in the chosen vacuum in which N (s)quark flavors condense, i.e. $r = N$) and $\mathcal{N} = (2, 2)$ supersymmetric $\text{CP}(N-1)$ model. This is referred to as the 2D-4D correspondence. This coincidence was first observed in [14] and then explained using the picture [4, 5] of monopoles confined to non-Abelian strings.³

In this paper we extend this 2D-4D correspondence to other vacua of $\mathcal{N} = 2$ supersymmetric QCD, namely, $r = N - 1$.

If the bulk $\mathcal{N} = 2$ theory is perturbed by a small mass term μ for the adjoint matter, the Coulomb branch is lifted and the theory has the so-called r vacua, where r is the number of condensed (s)quarks (in the large quark mass limit). The value of r cannot exceed the rank of the gauge group, i.e. $r \leq N$. If all quark masses are equal this deformation does not break $\mathcal{N} = 2$ supersymmetry and, in fact, reduces to the FI term to the leading order in μ in the $r = N$ vacuum [15, 16, 11]. In the $r = N - 1$ vacuum $N - 1$ (s)quarks and no monopoles condense. The absence of the condensed monopoles singles out $r = N - 1$. Below we assume that quark masses are generic so all r -vacua are isolated, no Higgs branches appear.

First, we obtain the classical theory on the non-Abelian string in the $r = N - 1$ vacuum. This is quite easy since it is given by a WCP model. Then we find a quantum deformation of the model using the method of resolvents suggested recently by Gaiotto, Gukov and Seiberg for surface defects [1].⁴ In much the same way as in the $r = N$ vacuum [11], the bulk monopoles are seen as kinks in the world-sheet theory. In the $\mu \rightarrow 0$ limit all vacua of the world-sheet theory are degenerate, and kinks are static. (If $\mu \neq 0$ the degeneracy is lifted, and strictly speaking there are no static kink solutions.)

Our calculation demonstrates the coincidence of 2D and 4D BPS spectra. The 2D-4D correspondence holds in the $r = N - 1$ vacua.

with N different elementary strings of the bulk theory.

³The important point here is that in the simplest version of this 2D-4D correspondence both BPS spectra do not depend on the FI parameter ξ [4, 5], for a more detailed discussion see Sec. 3. In fact, due to ξ independence, the 2D-4D correspondence can be interpreted as the coincidence between the BPS spectrum of the world-sheet $\text{CP}(N-1)$ model and that of the bulk theory taken at $\xi = 0$ (i.e. at a certain point on the Coulomb branch).

⁴Certain surface defects are related to non-Abelian strings in the low-energy limit. In our language the Gaiotto-Gukov-Seiberg setup can be understood as gauging of the flavor group and sending $\xi \rightarrow \infty$ [17]. In this limit all massive bulk states decouple, and non-Abelian strings become infinitely thin and infinitely heavy. In this paper we do not gauge the flavor group and consider finite values of ξ .

The paper is organized as follows. In Sec. 2 we briefly outline the structure of the r vacua in the μ -deformed $\mathcal{N} = 2$ QCD. In Sec. 3 we review the world sheet-theory and 2D-4D correspondence in these vacua. These two sections, Secs. 2 and 3, are needed to introduce relevant notation and specify our overall setting. Then we proceed to new results in the $r = N - 1$ vacuum. In Sec. 4 the effective theory on the non-Abelian string in the $r = N - 1$ vacuum is considered at the classical level. In Sec. 5 we study its quantum deformation. In Sec. 6 we prove the coincidence of the BPS spectra in the world-sheet and bulk theories. We then calculate the kink mass in the semiclassical approximation in the simplest $N = 2$ case. In Sec. 7 we discuss the μ -dependent deformation potential in the world-sheet theory while Sec. 8 summarizes our conclusions. In Appendices A and B we present semiclassical calculations of the roots of the Seiberg-Witten curve and the monopole mass in the $U(2)$ bulk theory, respectively.

2 r Vacua in $\mathcal{N} = 2$ QCD

2.1 μ -Deformed $\mathcal{N} = 2$ QCD

The gauge symmetry of our basic model is $U(N) = SU(N) \times U(1)$. In the absence of deformation the model under consideration is $\mathcal{N} = 2$ SQCD with N_f massive quark hypermultiplets. We assume that $N_f \geq N$ but $N_f < 2N$. The latter inequality ensures the theory to be asymptotically free.

In addition, we will introduce the mass term μ for the adjoint matter breaking $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$. Thus, the bulk theory is essentially the same as in [11]. The $\mathcal{N} = 2$ vector multiplet consists of the $U(1)$ gauge field A_μ and the $SU(N)$ gauge field A_μ^a , where $a = 1, \dots, N^2 - 1$, and their Weyl fermion superpartners plus complex scalar fields a , and a^a and their Weyl superpartners, respectively. The N_f quark multiplets of the $U(N)$ theory consist of the complex scalar fields q^{kA} and \tilde{q}_{Ak} (squarks) and their fermion superpartners — all in the fundamental representation of the $SU(N)$ gauge group. Here $k = 1, \dots, N$ is the color index while A is the flavor index, $A = 1, \dots, N_f$. We will treat q^{kA} and \tilde{q}_{Ak} as rectangular matrices with N rows and N_f columns.

Let us first discuss the undeformed $\mathcal{N} = 2$ theory. The superpotential is

$$\mathcal{W}_{\mathcal{N}=2} = \sqrt{2} \sum_{A=1}^{N_f} \left(\frac{1}{2} \tilde{q}_A \mathcal{A} q^A + \tilde{q}_A \mathcal{A}^a T^a q^A + m_A \tilde{q}_A q^A \right), \quad (2.1)$$

where \mathcal{A} and \mathcal{A}^a are chiral superfields, the $\mathcal{N} = 2$ superpartners of the gauge bosons of $U(1)$ and $SU(N)$, respectively. Then we add a mass term for the adjoint fields

$$\mathcal{W}_{\text{def}} = \mu \text{Tr} \Phi^2, \quad \Phi \equiv \frac{1}{2} \mathcal{A} + T^a \mathcal{A}^a \quad (2.2)$$

which breaks supersymmetry down to $\mathcal{N} = 1$, generally speaking. However, to the leading order in μ and if all quark masses are equal this term reduces to the Fayet-Iliopoulos F term which can be rotated into the D term [10]. The latter does not break $\mathcal{N} = 2$ supersymmetry [15, 16, 4, 11].

2.2 r Vacua

The r vacuum is a vacuum with r flavors of (s)quarks condensed. The r counting is assumed to be carried out in the weak coupling domain at large quark masses. It is obvious that the maximal value of r is N . The number of the isolated $r = N$ vacua is

$$\mathcal{N}_{r=N} = C_{N_f}^N = \frac{N_f!}{N!(N_f - N)!}, \quad (2.3)$$

see [8]. All gauge bosons are completely Higgsed, and the theory is in the color-flavor locked phase (assuming the quark masses to be close to each other). The (s)quark vacuum expectation values (VEVs) are determined by

$$\xi_P \sim \mu m_P, \quad P = 1, \dots, N. \quad (2.4)$$

A more precise definition of the set of the parameters ξ^P will be given below, see Eqs. (2.8), (2.9) and (2.14). For large values of ξ the bulk theory is at weak coupling. Then it can be studied semiclassically. In particular, non-Abelian strings confining monopoles are known to exist [2, 3, 4, 5].

For generic m_A the number of the isolated r vacua with $r < N$ is [18]

$$\mathcal{N}_{r < N} = \sum_{r=0}^{N-1} (N - r) C_{N_f}^r = \sum_{r=0}^{N-1} (N - r) \frac{N_f!}{r!(N_f - r)!} \quad (2.5)$$

representing the number of choices one can pick up r condensing quarks out of N_f quarks times the Witten index in the classically unbroken $SU(N-r)$ pure gauge theory.

Consider a particular vacuum in which the first r quarks develop non-vanishing VEVs. Quasiclassically, at large masses, the adjoint scalar VEVs are

$$\Phi^{\text{cl}} = -\frac{1}{\sqrt{2}} \text{diag} [m_1, \dots, m_r, 0, \dots, 0]. \quad (2.6)$$

The last $(N-r)$ entries vanish at the classical level. For those quarks which condense the corresponding eigenvalue of Φ is determined by the mass of this quark, while for those quarks which do not condense the corresponding eigenvalue should be a critical point of the deformation superpotential (2.2), which vanishes.

At the quantum level these zero entries acquire values determined by Λ , where Λ is the scale of $\mathcal{N} = 2$ QCD. In the classically unbroken $U(N-r)$ pure gauge sector the gauge symmetry gets broken through the Seiberg–Witten mechanism [19, 20]: first down to $U(1)^{N-r}$ and then almost completely by condensation of $(N-r-1)$ monopoles. A single $U(1)$ gauge factor survives.

This unbroken $U(1)$ factor in all $r < N$ vacua makes them critically different from the $r = N$ vacuum: in the latter there are no long-range forces.

Consider the non-Abelian limit when quark mass differences $\Delta m_{AB} = m_A - m_B$ are small, $\Delta m_{AB} \ll m_A$. The low-energy theory in the r vacuum has the gauge group

$$U(r) \times U(1)^{N-r}, \quad (2.7)$$

with N_f quark flavors charged under the $U(r)$ factor and $(N-r-1)$ monopoles charged under the $U(1)$ factors.

For $r > N_f/2$ and large $\xi \sim \mu m$ the $SU(r)$ non-Abelian quark sector is at weak coupling since it is asymptotically free.⁵ The quark condensates can be read-off from the superpotentials (2.1) and (2.2) using (2.6). They are

$$\begin{aligned} \langle q^{kA} \rangle &= \langle \bar{q}^{kA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\xi_1} & \dots & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \sqrt{\xi_r} & 0 & \dots & 0 \end{pmatrix}, \\ k &= 1, \dots, r, \quad A = 1, \dots, N_f, \end{aligned} \quad (2.8)$$

⁵The opposite case $r < N_f/2$ is discussed in [21].

with all other components vanishing. The first r parameters ξ in the quasi-classical approximation are

$$\xi_P \approx 2 \mu m_P, \quad P = 1, \dots, r. \quad (2.9)$$

These parameters can be made large in the large m_A limit even if μ is small.

2.3 Quantum effects

In quantum theory all parameters ξ_P are determined by the roots of the Seiberg-Witten (SW) curve [11, 22, 21] which in the case at hand takes the form [23]

$$y^2 = \prod_{P=1}^N (x - \phi_P)^2 - 4 \left(\frac{\Lambda_{N=2}}{\sqrt{2}} \right)^{2N-N_f} \prod_{A=1}^{N_f} \left(x + \frac{m_A}{\sqrt{2}} \right). \quad (2.10)$$

Here ϕ_P are gauge invariant parameters on the Coulomb branch. Instead of (2.6) one must write

$$\Phi \approx \text{diag} [\phi_1, \dots, \phi_N], \quad (2.11)$$

where

$$\phi_P \approx -\frac{m_P}{\sqrt{2}}, \quad P = 1, \dots, r; \quad \phi_P \sim \Lambda, \quad P = r+1, \dots, N. \quad (2.12)$$

In the $r = N$ vacuum the curve (2.10) has N double roots associated with condensation of $r = N$ quarks and reduces to

$$y^2 = \prod_{P=1}^N (x - e_P)^2, \quad (2.13)$$

where quasiclassically (at large masses) e_P 's are given by the mass parameters,

$$\sqrt{2}e_P \approx -m_P, \quad P = 1, \dots, N.$$

In fact, the (s)quark condensates in the $r = N$ vacuum are determined by the exact formula [11]

$$\xi_P = -2\sqrt{2}\mu e_P. \quad (2.14)$$

Now, consider the $r < N$ vacua. As was mentioned, in this paper we will focus on the simplest example of the $r < N$ vacuum, namely, $r = N - 1$. In

this vacuum $r = N - 1$ quarks condense in the large m_A limit, and there are no light (condensed) monopoles. To identify the $r = N - 1$ vacuum in terms of the curve (2.10) it is necessary to find such values of ϕ_P which ensure the Seiberg-Witten curve to have $N - 1$ double roots with ϕ_P approximately determined by the quark masses, see (2.12).

We know that the Seiberg-Witten curve factorizes [24],

$$y^2 = \prod_{P=1}^{N-1} (x - e_P)^2 (x - e_N^+) (x - e_N^-). \quad (2.15)$$

The first $r = N - 1$ double roots are determined by the mass parameters in the large mass limit, $\sqrt{2}e_P \approx -m_P$, $P = 1, \dots, (N - 1)$. The last two roots are much smaller and are determined by Λ . For the single-trace deformation superpotential (2.2) their sum vanishes [24],

$$e_N^+ + e_N^- = 0. \quad (2.16)$$

The root e_N^+ determines the value of the gaugino condensate [25],

$$(e_N^\pm)^2 = \frac{2S}{\mu}, \quad S = \frac{1}{32\pi^2} \langle \text{Tr } W_\alpha W^\alpha \rangle, \quad (2.17)$$

where the superfield W_α includes the gauge field strength tensor.

In terms of the roots of the Seiberg-Witten curve the quark VEVs are given by the formula⁶ [22, 21]

$$\xi_P = -2\sqrt{2}\mu \sqrt{(e_P - e_N^+)(e_P - e_N^-)} \quad (2.18)$$

for $P = 1, \dots, (N - 1)$.

2.4 Non-Abelian strings

As was already mentioned, our theory supports non-Abelian strings [2, 3, 4, 5]. At $\mu \ll (m_A, \Lambda)$ these strings are BPS-saturated [15, 16] and their tensions are determined exactly by the ξ parameters, namely [8, 11]

$$T_P = 2\pi|\xi_P|, \quad (2.19)$$

⁶In fact Eq. (2.18) is very general and determines the condensed state VEVs, namely, quarks or monopoles, independently of their nature in any vacuum with $r < N$ [21]. In the particular $r = N - 1$ vacuum which we consider here there are no condensed monopoles.

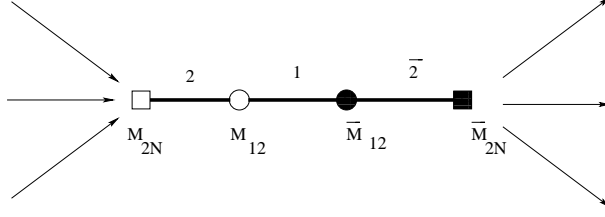


Figure 1: Example of the dipole meson formed as a result of breaking of the “second” string by pair creation of the monopole M_{2N} (shown by boxes) interpolating between the “second” string and the would-be N -th string (which is absent). Arrows denote unconfined flux. Circles denote monopoles $M_{PP'}$, $P, P' = 1, \dots, (N-1)$. Open and closed circles/boxes denote monopoles and antimonopoles, respectively.

with ξ_P given by (2.14) and (2.18) in the $r = N$ and $r < N$ vacua, respectively.

Both in the $r = N$ and $r = N - 1$ vacua non-Abelian strings are magnetic and confine monopoles. More precisely, the elementary monopole $M_{P,P+1}$ at $\mu \neq 0$ becomes the junction of P -th and $(P + 1)$ -th elementary non-Abelian strings, $P = 1, \dots, (N - 1)$.

In the $r = N - 1$ vacuum there is a peculiar feature distinguishing it from the $r = N$ vacuum. One string (say, the N -th string) is absent because the N -th quark does not condense. As a result, all strings become metastable. They can be broken by a pair creation of particular monopoles which interpolate between the P -th string and the would-be N -th string, which is in fact absent ($P = 1, \dots, (N - 1)$). An example of the monopole meson emerging in this way is shown in Fig. 1.

The endpoints emit fluxes of the unbroken $U(1)$ gauge field. This makes this meson a dipole-like configuration. Note that the non-Abelian fluxes of the $SU(N - 1)$ gauge group are always trapped and squeezed in the non-Abelian strings. Long-range forces are associated only with the unbroken $U(1)$ gauge factor.

In the large mass limit the masses of the monopoles which break the string become large, of the order of m/g^2 (where g^2 is the coupling constant in the $SU(N)$ group), and strings become metastable.

3 2D-4D correspondence in the $r = N$ vacuum

In this section we will briefly review non-Abelian strings in the $r = N$ vacuum and associated 2D-4D correspondence. We will start with the simplest version of the bulk theory with the FI D -term and then pass to μ -deformed $\mathcal{N} = 2$ QCD.

3.1 Bulk theory with the FI term

Consider $\mathcal{N} = 2$ QCD with the FI term of the D type. For simplicity we will assume now that $N_f = N$. As was already mentioned, the dynamics of orientational zero modes of non-Abelian string which become orientational moduli fields on the world sheet is described by two-dimensional $\mathcal{N} = (2, 2)$ supersymmetric $\text{CP}(N - 1)$ model, see e.g. [8] for a review. This model can be nicely written as a $\text{U}(1)$ gauge theory in the strong coupling limit (the so-called gauged formulation) [26]. The bosonic part of the action is

$$S_{\text{CP}(N-1)} = \int d^2x \left\{ |\nabla_\alpha n^P|^2 + \frac{1}{4e^2} F_{\alpha\beta}^2 + \frac{1}{e^2} |\partial_\alpha \sigma|^2 + |\sigma + m_P|^2 |n^P|^2 + \frac{e^2}{2} (|n^P|^2 - 2\beta)^2 \right\}, \quad (3.1)$$

where n^P are complex fields, $P = 1, \dots, N$,

$$\nabla_\alpha = \partial_\alpha - iA_\alpha,$$

σ is a complex scalar field, and summation over P is implied. The condition

$$|n^P|^2 = 2\beta \quad (3.2)$$

is implemented in the limit $e^2 \rightarrow \infty$. Moreover, in this limit the gauge field A_α and its $\mathcal{N} = 2$ bosonic superpartner σ become auxiliary and can be eliminated by virtue of the equations of motion.

In the limit of equal quark masses the global $\text{SU}(N)_{C+F}$ symmetry is unbroken, and strings are fully non-Abelian. This is a strong coupling quantum regime in the $\text{CP}(N - 1)$ model (3.1). The vector n^P is smeared all over the

entire $\text{CP}(N - 1)$ space due to quantum fluctuations and its average value vanishes [26]. The world-sheet theory develops a mass gap $\Lambda \ll \sqrt{\xi}$.

At small nonvanishing $|m_P - m_{P'}|$ the global $\text{SU}(N)_{C+F}$ symmetry is explicitly broken down to $U(1)^{(N-1)}$. A shallow potential is generated on the $\text{CP}(N - 1)$ moduli space as is seen from (3.1). As we increase $|m_P - m_{P'}|$ the strings become “more Abelian” and eventually evolve into Abelian Z_N strings, which correspond to N classical vacua of the world-sheet model (3.1)

$$n^P = \sqrt{2\beta} \delta^{PP_0}, \quad \sigma = -m_{P_0}, \quad (3.3)$$

where P_0 can take any of N values, $P_0 = 1, \dots, N$, see the review [8].

The two-dimensional coupling constant β ($\beta = 1/g^2$) is determined by the four-dimensional non-Abelian coupling g via the relation

$$\beta = \frac{2\pi}{g_{4D}^2}. \quad (3.4)$$

This relation is valid at the inverse transverse size of the string given by $g\sqrt{\xi}$ which plays the role of ultra-violet cutoff of the effective theory (3.1) on the string, see the review [8]. Given that β -functions of the bulk and world-sheet theories are the same this leads to the following identification

$$\Lambda_{2D} = \Lambda_{4D}, \quad (3.5)$$

which plays an important role in the coincidence of the BPS spectra of two theories.

3.2 More flavors

Adding “extra” quark flavors with degenerate masses we increase N_f from N up to a certain value $N_f > N$. The strings emerging in such theory are semilocal. In particular, the string solutions on the Higgs branches (typical for multiflavor theories) usually are not fixed-radius strings, but, rather, possess radial moduli, also known as the size moduli, see [27] for a comprehensive review of the Abelian semilocal strings. The transverse size of such strings is not fixed.

Non-Abelian semilocal strings in $\mathcal{N} = 2$ SQCD with $N_f > N$ were studied in [2, 5, 28, 29, 12]. The orientational moduli of the semilocal non-Abelian string can be described by a complex vector n^P (here $P = 1, \dots, N$),

while its $\tilde{N} = (N_f - N)$ size moduli are parametrized by a complex vector ρ^K ($K = N + 1, \dots, N_f$). The effective two-dimensional theory which describes the internal dynamics of the non-Abelian semilocal string is the $\mathcal{N} = (2, 2)$ weighted CP model, which includes both types of fields. The bosonic part of the action in the gauged formulation (which assumes taking the limit $e^2 \rightarrow \infty$) has the form

$$\begin{aligned}
S_{\text{WCP}} = \int d^2x \left\{ |\nabla_\alpha n^P|^2 + |\tilde{\nabla}_\alpha \rho^K|^2 + \frac{1}{4e^2} F_{\alpha\beta}^2 + \frac{1}{e^2} |\partial_\alpha \sigma|^2 \right. \\
+ |\sigma + m_P|^2 |n^P|^2 + |\sigma + m_K|^2 |\rho^K|^2 + \frac{e^2}{2} (|n^P|^2 - |\rho^K|^2 - 2\beta)^2 \Big\}, \\
P = 1, \dots, N, \quad K = N + 1, \dots, N_f.
\end{aligned} \tag{3.6}$$

The fields n^P and ρ^K have charges $+1$ and -1 with respect to the auxiliary U(1) gauge field;⁷ hence, the corresponding covariant derivative $\tilde{\nabla}$ in (3.6) is

$$\tilde{\nabla}_\alpha = \partial_\alpha + iA_\alpha.$$

As in the CP($N - 1$) model, small mass differences $|m_A - m_B|$ lift orientational and size zero modes generating a shallow potential on the modular space.

The coupling constant β in (3.6) is related to the bulk coupling via (3.4) which ensures the coincidence of scales of bulk and world sheet theory, see (3.5), where we used that the first coefficient of the β function $b = 2N - N_f$ is the same for the bulk and world-sheet theories.

3.3 Exact superpotential

An exact twisted superpotential of the Veneziano-Yankielowicz type [30] is known in the CP($N - 1$) model [31, 32, 33, 14]. This superpotential was later

⁷In fact, the theory on the semilocal non-Abelian string is not exactly the WCP model (3.6). The actual theory that emerges on the world sheet was called the zn model [12, 13]. It has a somewhat different metric of the target space. The WCP model (3.6) correctly reproduce the BPS spectrum of the world-sheet theory, which comes out exactly the same as in the zn model [12, 13]. In what follows we use a similar approach to describe the BPS spectrum of the 2D theory on the non-Abelian string in the $r = N - 1$ vacuum.

generalized to the case of the WCP models in [34, 35]. Integrating out the fields n^P and ρ^K we obtain the following exact twisted superpotential:

$$\mathcal{W}_{\text{WCP}}(\sigma) = \frac{1}{4\pi} \left\{ \sum_{P=1}^N (\sigma + m_P) \ln \frac{\sigma + m_P}{\Lambda} - \sum_{K=N+1}^{N_F} (\sigma + m_K) \ln \frac{\sigma + m_K}{\Lambda} - (N - \tilde{N}) \sigma \right\}, \quad (3.7)$$

where we use one and the same notation σ for the twisted superfield [33] and its lowest scalar component. Minimizing this superpotential with respect to σ we get the equation for the VEVs of σ (the so-called twisted chiral ring equation),

$$\prod_{P=1}^N (\sigma + m_P) = \Lambda^{(N-\tilde{N})} \prod_{K=N+1}^{N_f} (\sigma + m_K). \quad (3.8)$$

The masses of the BPS kinks interpolating between the vacua σ_P and $\sigma_{P'}$ are given by the appropriate differences of the superpotential (3.7) calculated at distinct roots [34, 14, 35],

$$M_{PP'}^{\text{BPS}} = 2 |\mathcal{W}_{\text{WCP}}(\sigma_{P'}) - \mathcal{W}_{\text{WCP}}(\sigma_P)|, \quad P, P' = 1, \dots, N. \quad (3.9)$$

Due to the presence of branches in the logarithmic functions in (3.7) each kink come together with a tower of dyonic kinks carrying global U(1) charges (for more details see e.g. [36]). In addition to kinks the BPS spectrum of the model contains elementary excitations with masses given by $|m_A - m_P|$, $A = 1, \dots, N_f$, $P = 1, \dots, N$.

The masses obtained from (3.9) were shown to coincide with those of the monopoles and dyons in the bulk theory. The latter are given by the period integrals of the Seiberg–Witten curve (2.10).

As was mentioned in Sec. 1, this coincidence was observed in [14, 35] and explained later in [4, 5] using the picture of confined bulk monopoles which are seen as kinks in the world sheet theory. A crucial point is that both monopoles and kinks are BPS-saturated states⁸, and their masses cannot depend on the non-holomorphic parameter ξ [4, 5]. This means that,

⁸Confined monopoles, being junctions of two distinct 1/2-BPS strings, are 1/4-BPS states in the bulk theory [4].

although confined monopoles look physically very different from unconfined monopoles on the Coulomb branch of the bulk theory (in the particular singular point which becomes the $r = N$ vacuum at nonzero ξ), their masses are the same. Moreover, they coincide with the masses of kinks in the world-sheet theory.

Note that the roots of the vacuum equation (3.8) coincide with the double roots of the Seiberg–Witten curve (2.10) of the bulk theory [14, 35],

$$\sigma_P = \sqrt{2} e_P. \quad (3.10)$$

This is the key technical reason which leads to the coincidence of the BPS spectra.

3.4 The $r = N$ vacuum in μ -deformed $\mathcal{N} = 2$ QCD

Now let us switch off the Fayet-Iliopoulos D term in the bulk theory and consider instead the F term deformation (2.2). In [11, 37] it was shown that at generic quark masses $\mathcal{N} = (2, 2)$ supersymmetry is broken down to $\mathcal{N} = (0, 2)$ even to the leading order in μ . For the single-trace deformation (2.2) the bosonic part of the low-energy world-sheet theory becomes

$$S_{2D} = S_{(2,2)} + \int d^2x V_{\text{def}}(\sigma), \quad (3.11)$$

where $S_{(2,2)}$ is the action of $\mathcal{N} = (2, 2)$ supersymmetric model (3.6) while the deformation potential is given by

$$V_{\text{def}}(\sigma) = 4\pi |\mu\sigma|. \quad (3.12)$$

The total scalar potential given by the sum of the twisted mass potential in (3.6) and deformation (3.12) is schematically shown in Fig. 2 a. Its N minima correspond to tensions of N elementary non-Abelian strings,

$$V(\sigma_P) = T_P, \quad P = 1, \dots, N. \quad (3.13)$$

To see this we note that the vacuum values σ_P are still given by solutions of the chiral ring equation (3.8) corresponding to the limit $\mu \rightarrow 0$. Then the coincidence of the roots of the bulk and world-sheet theory (3.10), together with Eqs. (2.14) and (2.19), gives (3.13). At $\mu \neq 0$ and generic masses the

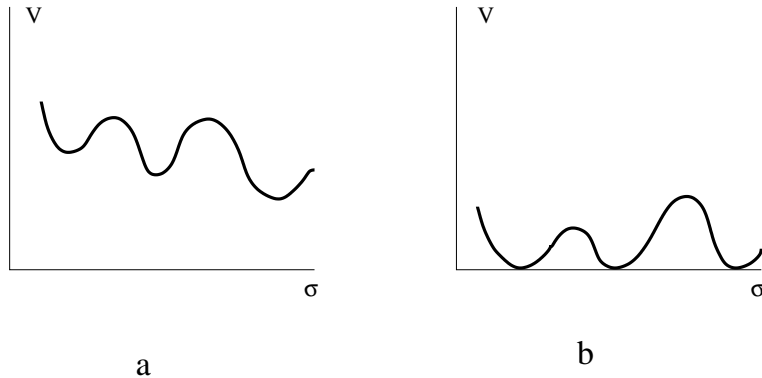


Figure 2: a. Schematic picture of the scalar potential in the theory (3.11). The complex variable σ is schematically represented by the horizontal axis. Minima of the potential correspond to elementary non-Abelian strings with tensions given by (3.13). b. The same potential in the limit $\mu = 0$.

minima are non-degenerate, only the lowest lying vacuum is stable, no static kink solutions exist.

The stability of the lowest vacuum in two dimensions means that the lightest of N non-Abelian strings is stable, others become metastable. Moreover, since generically string tensions do not vanish, $\mathcal{N} = (0, 2)$ supersymmetry is broken spontaneously already at the classical level [11]. The barriers between different vacua of the potential in Fig. 2 are of the order of the quark mass differences $|m_A - m_B|^2$.

The 2D-4D correspondence manifests itself as follows. Both the confined bulk monopoles and kinks of the world-sheet theory are no longer BPS-saturated, and we cannot expect that their masses are independent of μ (or, which is the same, ξ in the case at hand). We expect that the kink mass is

$$M_{PP'}^{\text{kink}} = 2 |\mathcal{W}(\sigma_{P'}) - \mathcal{W}(\sigma_P)| + O(\mu), \quad P, P' = 1, \dots, N, \quad (3.14)$$

where $\mathcal{W}(\sigma) = \mathcal{W}_{\text{WCP}}$ for the $r = N$ vacuum (see (3.7)) and the term $O(\mu)$ represents non-BPS μ -corrections. Strictly speaking, the kink is not defined as a static object at $\mu \neq 0$. Because of the difference of tensions it must accelerate.

The mass of the confined monopole is given by

$$M_{PP'}^{\text{monopole}} = \left| \frac{\sqrt{2}}{2\pi i} \oint_{\beta_{PP'}} d\lambda_{SW} \right| + O(\mu), \quad P, P' = 1, \dots, N, \quad (3.15)$$

where integral of the Seiberg-Witten differential [19, 20, 38, 39, 40, 41] goes along the β contour through shrinking cuts associated with double roots e_P and $e_{P'}$. The second term represents non-BPS μ -corrections.

Since the kink in the low-energy theory on the non-Abelian string represents a confined bulk monopole their masses should be the same,

$$M_{PP'}^{\text{monopole}} = M_{PP'}^{\text{kink}}, \quad P, P' = 1, \dots, N, \quad \mu \rightarrow 0. \quad (3.16)$$

In other words, the BPS spectra of the theory with the action $S_{(2,2)}$ and the bulk theory on the Coulomb branch (at the particular singular point which becomes the $r = N$ vacuum upon μ -deformation) should coincide with each other. As was already explained, this conclusion was explicitly checked in [14, 35] for the $r = N$ vacuum. In this form the 2D-4D correspondence is easy to generalize to other r vacua. We will use this form of the 2D-4D correspondence in what follows.

It seems somewhat confusing to take limit $\mu \rightarrow 0$ in the world-sheet theory because in this limit strings disappear while monopoles become unconfined. However, if we forget that this theory represents the theory on the string and view it as a 2D theory *per se*, we see from Eq. (3.11) that this limit is perfectly well defined. Moreover, both kinks and monopoles become BPS saturated in this limit. The potential of the 2D theory in this limit is shown in Fig. 2 b.

4 Classical theory on the non-Abelian string in the $r = N - 1$ vacuum

Now we start construction of the world-sheet theory in the $r = N - 1$ vacuum at the classical level.

In much the same way as in the $r = N$ vacuum the low-energy theory on the string in the $r = N - 1$ vacuum is given by the sum of an $\mathcal{N} = (2, 2)$ supersymmetric theory (let us call it $T_{(2,2)}$) and a μ -deformation,

$$S_{2D} = S_{(2,2)} + \int d^2x V_{\text{def}}(\sigma), \quad (4.1)$$

where $S_{(2,2)}$ is the action of the $T_{(2,2)}$ theory. The deformation potential V_{def} at its minima gives the tensions of the non-Abelian strings, cf. (3.12). The

string tensions in the $r = N - 1$ vacua are given by

$$V_{\text{def}}(\sigma_P) = T_P = 4\pi\sqrt{2} \left| \mu \sqrt{(e_P - e_N^+)(e_P - e_N^-)} \right| \quad (4.2)$$

for $P = 1, \dots, (N - 1)$, see Eqs. (2.18) and (2.19).

Now we have only $(N - 1)$ strings, while the N -th string is absent. The associated minimum of V_{def} is the ground state at zero energy.

Let us find $\mathcal{N} = (2, 2)$ supersymmetric theory $T_{(2,2)}$ neglecting for a while the deformation potential V_{def} in (4.1). We will discuss it later in Sec. 7. We also assume for simplicity that $N_f = N$.

Consider first the quasiclassical limit

$$m_A \gg \Lambda, \quad \Delta m_{AB} = (m_A - m_B) \ll m_A. \quad (4.3)$$

In this limit the low-energy gauge group of the bulk theory becomes

$$U(N - 1) \times U(1), \quad (4.4)$$

where the $U(1)$ factor is unbroken, and the theory has $N_f = N$ quarks charged under the $U(N - 1)$ factor, see Eqs. (2.6) and (2.7). The (s)quark fields develop VEVs given by Eq. (2.8) with $r = N - 1$.

Thus, this low-energy theory supports non-Abelian strings. Since the number of the quark flavors N is larger than the rank $r = N - 1$ of the low-energy gauge group by 1, these strings are semilocal. The world-sheet theory is given by the WCP model⁹ (3.6) with $N - 1$ orientational moduli n^P with charge $+1$ ($P = 1, \dots, N - 1$), plus a single size modulus ρ , with charge -1 . With the mass parameters chosen according to (4.3) $N - 1$ strings are (meta)stable.

The coefficient b of the β function of this low-energy world-sheet theory is the sum of charges of the n^P and ρ fields, namely $b_{LE} = (N - 1) - 1 = N - 2$. This coefficient coincides with the coefficient b of the bulk theory in the low-energy limit, $b_{LE} = 2(N - 1) - N_f = N - 2$.

Now, let us relax the condition $\Delta m_{AB} \ll m_A$. To determine the world-sheet theory $T_{(2,2)}$ we can use the following procedure.

Let us start from the $r = N$ vacuum where the theory on the non-Abelian string is given by the $CP(N - 1)$ model (3.1), with the β -function coefficient $b = N$. Then we reduce the mass of the N -th quark m_N . The point $m_N = 0$

⁹See footnote 7.

is a point where two vacua ($r = N$ and $r = N - 1$) coalesce. At this point we can “jump” into the $r = N - 1$ vacuum and then increase m_N to its initial value. In this process our world-sheet theory is smoothly deformed from the $\text{CP}(N - 1)$ model to the theory $T_{(2,2)}$ sought for.

This implies that the theory $T_{(2,2)}$ is given by the $\text{CP}(N - 1)$ model in which the “last” field n^N is taken with $m_N = 0$ plus an extra conformal sector which does not spoil the correct β function. The point is that the coefficient b of the β function of world-sheet theory should coincide with the one for the bulk theory, $b = N$.

Thus, the conformal sector must consist of two complex fields z and ρ , with charges $+1$ and -1 , respectively. At large masses in the limit (4.3) the n^N field present in the $\text{CP}(N - 1)$ model, as well as z , become massive and decouple, so we are left with the low-energy WCP model described after Eq. (4.4), see Eq. (4.5) with n^N and z crossed out.

Combined with $\mathcal{N} = (2, 2)$ supersymmetry this leads us to the following bosonic action of $T_{(2,2)}$:

$$\begin{aligned}
S_{(2,2)}^{\text{cl}} &= \int d^2x \left\{ |\nabla_\alpha n^P|^2 + |\nabla_\alpha n^N|^2 + |\tilde{\nabla}_\alpha \rho|^2 + |\nabla_\alpha z|^2 + \frac{1}{4e^2} F_{\alpha\beta}^2 + \frac{1}{e^2} |\partial_\alpha \sigma|^2 \right. \\
&+ |\sigma + m_P|^2 |n^P|^2 + |\sigma + m_N|^2 |\rho|^2 + |\sigma|^2 |n^N|^2 + |\sigma|^2 |z|^2 \\
&+ \left. \frac{e^2}{2} (|n^P|^2 + |n^N|^2 + |z|^2 - |\rho|^2 - 2\beta)^2 \right\}, \\
P &= 1, \dots, N - 1.
\end{aligned} \tag{4.5}$$

The physical meaning of the n^N and z fields is related to “unwinding” of the $(N - 1)$ -th string into the N -th string which is, in fact, absent, see Sec. 2.4 and Fig. 1. The coefficient b of this WCP model is equal to the sum of the charges of all charged fields,

$$b = (N - 1) - 1 + 1 + 1 = N, \tag{4.6}$$

where the first contribution comes from $(N - 1)$ n^P fields, the second one comes from the size modulus ρ and the last two contributions come from the “unwinding” fields n^N and z .

In Sec. 5 we will see that (4.5) is actually the action of our 2D theory at the *classical level*. At the quantum level the model will be modified by bulk quantum corrections. The superscript in $S_{(2,2)}^{\text{cl}}$ reflects this.

As a check let us choose one of $(N - 1)$ vacua of the theory (4.5),

$$n^P = \sqrt{2\beta} \delta^{PP_0}, \quad \sigma = -m_{P_0}, \quad P = 1, \dots, N - 1, \quad (4.7)$$

where P_0 can be chosen arbitrarily from the set $\{1, \dots, N - 1\}$. Then in the quasiclassical limit (4.3) the fields n^P with $P \neq P_0$ have masses

$$|m_P - m_{P_0}|,$$

the field ρ has mass $|m_N - m_{P_0}|$, while the fields n^N and z are much heavier, their mass is $|m_{P_0}|$. Therefore n^N and z can be integrated out which leads us to the low-energy world-sheet theory which describes non-Abelian strings in the bulk theory with the gauge group (4.4) in the quasiclassical approximation (4.3).

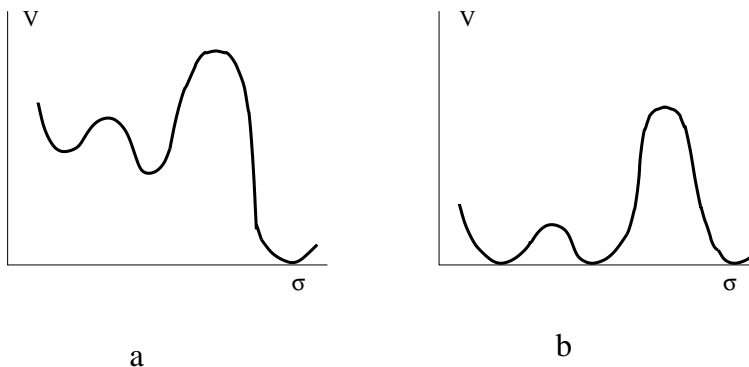


Figure 3: a. Schematic picture of the scalar potential in the theory (4.1) in the quasiclassical limit (4.3). Now the “last” vacuum has zero energy reflecting the absence of the N -th string. b. The same potential in the limit $\mu = 0$.

Qualitative behavior of the total scalar potential in (4.1) is shown in Fig. 3a. Barriers between different strings are described by the scalar potential in (4.5). The heights of these barriers are of the order of the quark mass differences squared, $|m_P - m_{P+1}|^2$. The height of the last barrier associated with the metastability of the $(N - 1)$ -th string is of the order of $|m_{N-1}|^2$. As was already mentioned, the vacuum energies at the minima are proportional to μ and are given by string tensions in Eq. (4.2). (They are not reflected in (4.5).) The last N -th “vacuum” at $\sigma_N = 0$ has zero energy reflecting the absence of the N -th string. Figure 3b shows the same potential in the limit of $\mu = 0$. All vacua become stable and the BPS-saturated kinks become well defined.

To conclude this section let us discuss a more general setup with $N_f > N$, i.e. we will add more quark flavors in the bulk theory. On the string world sheet this leads to emergence of extra size moduli ρ . The theory $T_{(2,2)}$ is still given by the WCP model similar to the one in (4.5) with $N - 1$ orientational moduli n^P with charge $+1$ ($P = 1, \dots, N - 1$) and $(N_f - N + 1)$ size moduli ρ^K , with charge -1 ($K = N, \dots, N_f$) plus two fields n^N and z with charges $+1$. The first coefficient of the β function of this world sheet theory

$$b = (N - 1) - (N_f - N + 1) + 1 + 1 = 2N - N_f \quad (4.8)$$

coincides with the coefficient in the bulk theory.

5 Quantum deformation

5.1 Superpotential with quantum corrections

The exact twisted superpotential for the classical version of the theory $T_{(2,2)}$ (see (4.5)) is

$$\begin{aligned} \mathcal{W}^{\text{cl}} = & \frac{1}{4\pi} \left\{ \sum_{P=1}^{N-1} (\sigma + m_P) \ln \frac{\sigma + m_P}{e\Lambda} \right. \\ & \left. + 2\sigma \ln \frac{\sigma}{e\Lambda} - \sum_{K=N}^{N_f} (\sigma + m_K) \ln \frac{\sigma + m_K}{e\Lambda} \right\}, \end{aligned} \quad (5.1)$$

where the first term comes from integrating out $(N - 1)$ orientational moduli n^P , the second term comes from the “unwinding” fields n^N and z and the last term comes from $N_f - N + 1$ size moduli ρ^K . Here we generalize (4.5) adding more flavors, so that $N_f \geq N$, see the discussion at the end of the previous section.

We can rewrite (5.1) identically in the form

$$\begin{aligned} \mathcal{W}^{\text{cl}}(\sigma) = & \frac{1}{4\pi} \left\{ 2 \text{Tr} \left[(\sigma - \sqrt{2} \Phi^{\text{cl}}) \ln \frac{\sigma - \sqrt{2} \Phi^{\text{cl}}}{e\Lambda} \right] \right. \\ & \left. - \sum_{A=1}^{N_f} (\sigma + m_A) \ln \frac{\sigma + m_A}{e\Lambda} \right\}, \end{aligned} \quad (5.2)$$

where

$$\Phi^{\text{cl}} = -\frac{1}{\sqrt{2}} \begin{pmatrix} m_1 & \dots & 0 & 0 \\ \dots & \dots & \dots & 0 \\ 0 & \dots & m_{N-1} & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix}, \quad (5.3)$$

see (2.6) with $r = N - 1$.

Below we will prove that, unlike the $r = N$ vacuum [11], nonperturbative effects in the bulk (in the form of the gluino condensate) affect the target space of the world-sheet model in the case $r = N - 1$. In the former case, $r = N$, the gluino condensate vanishes.

In the spirit of the approach put forward recently by Gaiotto, Gukov and Seiberg [1] we argue that the exact twisted superpotential of the world-sheet theory $T_{(2,2)}$ is

$$\begin{aligned} \mathcal{W}(\sigma) = & \frac{1}{4\pi} \left\{ 2 \left\langle \text{Tr} \left[(\sigma - \sqrt{2}\Phi) \ln \frac{\sigma - \sqrt{2}\Phi}{e\Lambda} \right] \right\rangle \right. \\ & \left. - \sum_{A=1}^{N_f} (\sigma + m_A) \ln \frac{\sigma + m_A}{e\Lambda} \right\}, \end{aligned} \quad (5.4)$$

where the braces imply that the quantum average is taken over the bulk theory.

Following [1] we calculate the second derivative of the quantum average in (5.4) with respect to σ to obtain the resolvent

$$T(\sigma) = \left\langle \text{Tr} \frac{1}{\sigma - \sqrt{2}\Phi} \right\rangle. \quad (5.5)$$

The exact solution for this object was found by Cachazo, Seiberg and Witten [25] precisely in our bulk theory. In particular, for the bulk deformation (2.2) in the r vacuum we have

$$\begin{aligned} T(\sigma)_r = & \frac{1}{2} \sum_{A=1}^{N_f} \frac{1}{\sigma + m_A} + \frac{1}{2} \frac{2N - N_f}{\sqrt{\sigma^2 - \frac{4S}{\mu}}} \\ & - \frac{1}{2} \sum_{A=1}^r \frac{\sqrt{m_A^2 - \frac{4S}{\mu}}}{\sqrt{\sigma^2 - \frac{4S}{\mu}}(\sigma + m_A)} + \frac{1}{2} \sum_{A=r+1}^{N_f} \frac{\sqrt{m_A^2 - \frac{4S}{\mu}}}{\sqrt{\sigma^2 - \frac{4S}{\mu}}(\sigma + m_A)}, \end{aligned} \quad (5.6)$$

where S is the gaugino condensate. Note that the ratio S/μ depends only on masses and Λ , it does not depend on μ , see (2.17) demonstrating that $S \propto \mu$.

In the $r = N$ vacuum the gaugino condensate is zero, $S = 0$. In this case the resolvent $T(\sigma)$ in (5.6) reduces to its classical expression

$$T(\sigma)_{r=N} = \sum_{P=1}^N \frac{1}{\sigma + m_P}. \quad (5.7)$$

This gives the superpotential (3.7) for the theory on non-Abelian string in the $r = N$ vacuum. Quantum deformation is absent in this case. Since the superpotential (3.7) contains only one-loop logarithmic terms we can say that bulk instantons do not penetrate on the world sheet in the $r = N$ vacuum.

Consider now the $r = N - 1$ vacuum. Integrating over σ in (5.6) and substituting the result into (5.4) we get

$$\begin{aligned} \partial_\sigma \mathcal{W}(\sigma) = & \frac{1}{4\pi} \left\{ \sum_{A=1}^{N-1} \ln \frac{(\sigma + m_A)}{\Lambda} - \sum_{A=N}^{N_f} \ln \frac{(\sigma + m_A)}{\Lambda} \right. \\ & \left. + (2N - N_f) \ln \frac{t}{\Lambda} - \sum_{A=1}^{N-1} \ln \frac{t_A}{\Lambda} + \sum_{A=N}^{N_f} \ln \frac{t_A}{\Lambda} \right\}, \end{aligned} \quad (5.8)$$

where we define

$$t = \frac{1}{2} \left(\sigma + \sqrt{\sigma^2 - \frac{4S}{\mu}} \right) \quad (5.9)$$

and

$$t_A = \frac{1}{2} \left(\sqrt{\sigma^2 - \frac{4S}{\mu}} + \frac{\sigma + \frac{4S}{\mu m_A}}{\sqrt{1 - \frac{4S}{\mu m_A^2}}} \right). \quad (5.10)$$

Equation (5.8) is our final result for the exact twisted superpotential of the theory $T_{(2,2)}$ on the non-Abelian string in the $r = N - 1$ vacuum. It takes into account the quantum deformation produced by the bulk instantons. The latter generate gaugino condensate which results in the emergence of the square root cut in the σ plane, see (5.9) and (5.10). The emergence of this cut is a response of the 2D world-sheet theory to the cut in the SW curve present in the bulk theory in the $r = N - 1$ vacuum, see (2.15).

If we neglect the gaugino condensate in the quasiclassical limit of large masses we get $t \approx t_A \approx \sigma$ returning us to the “classical” superpotential (5.1).

5.2 Chiral ring equation

The equation determining the vacuum values of σ is obtained by requiring $\partial_\sigma \mathcal{W}(\sigma) = 0$. Exponentiating (5.8) we obtain

$$t^{(2N-N_f)} \prod_{P=1}^{N-1} \frac{(\sigma + m_P)}{t_P} = \Lambda^{(2N-N_f)} \prod_{K=N}^{N_f} \frac{(\sigma + m_K)}{t_K}. \quad (5.11)$$

Let us first approximately solve this equation in the quasiclassical limit of large (generic) quark masses,

$$m_A \gg \Lambda, \quad \Delta m_{AB} \sim m_A. \quad (5.12)$$

To the leading order $N - 1$ roots are

$$\sigma_P \approx -m_P \quad P = 1, \dots, N - 1.$$

Consider the first correction to a particular root σ_{P_0} where P_0 is arbitrarily chosen from the set $\{1, \dots, N - 1\}$. From Eq. (5.11) we find

$$\sigma_{P_0} \approx -m_{P_0} + \frac{\Lambda^{(2N-N_f)}}{m_{P_0}^2} \frac{\prod_{K=N}^{N_f} (m_K - m_{P_0})}{\prod_{P \neq P_0} (m_P - m_{P_0})} + \dots, \quad (5.13)$$

where the product in the denominator runs over $P = 1, \dots, N - 1$ and we neglect the gaugino condensate as compared to the quark masses.

In Appendix A we present the calculation of the double roots of the SW curve in the $r = N - 1$ vacuum with the same accuracy. Comparing (5.13) and (A.3) we see that

$$\sigma_P = \sqrt{2} e_P, \quad P = 1, \dots, N - 1. \quad (5.14)$$

Now let us calculate “small” roots in (5.11). We will see that they will be of the order of $\sqrt{S/\mu} \sim e_N$, see (2.17). Therefore we still can neglect S/μ as compared to σ or m_A . This gives $t_A \approx t$, and then Eq. (5.11) reduces to

$$t^2 \prod_{P=1}^{N-1} (\sigma + m_P) = \Lambda^{(2N-N_f)} \prod_{K=N}^{N_f} (\sigma + m_K). \quad (5.15)$$

Neglecting small σ as compared to the quark masses we get

$$t^2 \approx \Lambda^{(2N-N_f)} \frac{\prod_{K=N}^{N_f} m_K}{\prod_{P=1}^{N-1} m_P}, \quad t \approx \pm \sqrt{\Lambda^{(2N-N_f)} \frac{\prod_{K=N}^{N_f} m_K}{\prod_{P=1}^{N-1} m_P}}. \quad (5.16)$$

The values of the unpaired roots of the SW curve e_N^\pm are calculated in Appendix A in the leading order. Equation (A.5) shows that the combination under the square root sign in (5.16) is exactly $e_N^2/2$. Then the above equation gives

$$2t = \left(\sigma + \sqrt{\sigma^2 - 2e_N^2} \right) = \sqrt{2} e_N^\pm, \quad (5.17)$$

where we use (5.9) and (2.17).

From this equation we find that two “small” roots σ^\pm are

$$\sigma_N^\pm \approx \pm 2 \sqrt{\Lambda^{2N-N_f} \frac{\prod_{K=N}^{N_f} m_K}{\prod_{P=1}^{N-1} m_P}}. \quad (5.18)$$

They are given by unpaired roots of the SW curve,

$$\sigma_N^\pm = \sqrt{2} e_N^\pm. \quad (5.19)$$

Thus, we see that VEVs of σ are given by the roots of the SW curve in the $r = N - 1$ vacuum in much the same way as is the case in the $r = N$ vacuum. We proved this statement in the quasiclassical approximation above. However, it is very likely that this relation is exact. We assume this conjecture to be true in what follows.

The emergence of two roots σ_N^\pm is a reflection of the cut in the σ plane. Classically the cut is invisible.

6 2D-4D correspondence in the $r = N - 1$ vacuum

Since confined monopoles are represented by kinks in the world-sheet theory their masses should coincide. In this section we explicitly confirm this expectation by verifying the equality in (3.16).

6.1 Kink masses versus monopole masses

If we neglect non-BPS μ -corrections in (3.14) the kink masses are

$$M_{PP'}^{\text{kink}} = 2 |\mathcal{W}(\sigma_{P'}) - \mathcal{W}(\sigma_P)|, \quad P, P' = 1, \dots, N, \quad (6.1)$$

where $\mathcal{W}(\sigma)$ for $T_{(2,2)}$ is determined by (5.8). Starting from $\partial_\sigma \mathcal{W}(\sigma)$ from (5.8) and integrating by parts we can present the kinks masses in the form

$$M_{PP'}^{\text{kink}} = \left| \frac{1}{\pi} \int_{\sigma_P}^{\sigma_{P'}} d\sigma \left\{ \frac{2N - N_f}{2} \frac{\sigma}{\sqrt{\sigma^2 - \frac{4S}{\mu}}} - \frac{1}{2} \sum_{A=1}^{N-1} \frac{\sigma \sqrt{m_A^2 - \frac{4S}{\mu}}}{\sqrt{\sigma^2 - \frac{4S}{\mu}} (\sigma + m_A)} + \frac{1}{2} \sum_{A=N}^{N_f} \frac{\sigma \sqrt{m_A^2 - \frac{4S}{\mu}}}{\sqrt{\sigma^2 - \frac{4S}{\mu}} (\sigma + m_A)} \right\} \right|, \quad (6.2)$$

where we drop the total derivative term. It is zero for the vacuum values of σ due to Eq. (5.11).

In Appendix B we present calculation of the monopole mass in the $r = N-1$ vacuum for the simplest example: U(2) theory with two flavors, $N_f = 2$. Taking $N = N_f = 2$ in (6.2) we see that the kink mass coincides with the mass (B.4) of the monopole. The important input here is the coincidence of the roots of the SW curve with the vacuum values of σ , see (5.14) and (5.19).

6.2 Kink mass in U(2)

For illustration we calculate the kink/monopole mass in the $r = N-1$ vacuum in the simplest U(2) theory with $N_f = 2$ in the semiclassical approximation

$$m_1 \sim m_2 \gg \Lambda. \quad (6.3)$$

In this case Eq. (5.8) reads

$$\partial_\sigma \mathcal{W}(\sigma) \approx \frac{1}{4\pi} \left\{ \ln \frac{(\sigma + m_1)}{\Lambda} - \ln \frac{(\sigma + m_2)}{\Lambda} + 2 \ln \frac{t}{\Lambda} \right\}, \quad (6.4)$$

where we used $t_A \approx t$ in the semiclassical approximation (6.3), see (5.9) and (5.10). Integrating over σ we get

$$M^{\text{kink}} \approx \left| \frac{1}{2\pi} \left\{ m_1 \ln(\sigma + m_1) - m_2 \ln(\sigma + m_2) - 2 \sqrt{\sigma^2 - 4\Lambda^2} \frac{m_2}{m_1} \right\} \right|_{\sigma_1}^{\sigma_2^\pm}, \quad (6.5)$$

Here we used (A.5) to calculate $4S/\mu$, see (2.17). In (6.5) the kink central charge is given by the difference of the expression in the braces calculated at $\sigma = \sigma_2^\pm$ and $\sigma = \sigma_1$, respectively. We drop the term proportional to σ since it vanishes due to (5.15). As usual, different branches of the logarithmic functions will give dyonic kinks.

Equations (5.13) and (5.18) imply

$$\sigma_1 \approx -m_1 + (m_2 - m_1) \frac{\Lambda^2}{m_1^2}, \quad \sigma_2^\pm \approx \pm 2\Lambda \sqrt{\frac{m_2}{m_1}}. \quad (6.6)$$

Substituting this in (6.5) we finally get

$$M^{\text{kink}} = \left| \frac{1}{\pi} \left\{ m_1 \ln \frac{m_1}{\Lambda} + \frac{m_1}{2} \ln \frac{m_1}{m_2 - m_1} - \frac{m_2}{2} \ln \frac{m_2}{m_2 - m_1} + m_1 + \dots \right\} \right|, \quad (6.7)$$

where the ellipses denote terms proportional to Λ . Note that the result for the kink mass does not depend on the particular choice of the upper limit¹⁰ either $\sigma = \sigma_2^+$ or $\sigma = \sigma_2^-$.

The coefficient in front of the logarithm of Λ here is $b/2\pi$; it reflects the correct coefficient of the β function, $b = 2$.

It is instructive to consider different limits in (6.7). First take the limit $m_2 \rightarrow \infty$ decoupling the second flavor. In this case the coefficient of the β function b becomes $b_1 = 3$ while the effective scale of the theory is $\Lambda_1^3 = m_2 \Lambda^2$. Eq. (6.7) gives in this limit

$$M_{N_f=1}^{\text{kink}} = \left| \frac{1}{\pi} \left\{ \frac{3}{2} m_1 \ln \frac{m_1}{\Lambda_1} + \frac{1}{2} m_1 + \dots \right\} \right|. \quad (6.8)$$

We see that the logarithmic term here has the correct coefficient $b_1/2\pi = 3/2\pi$ in front of $\ln \Lambda_1$.

Another interesting limit is that of the equal masses $\Delta m = m_2 - m_1 \rightarrow 0$. In this limit (6.7) reduces to

$$M^{\text{kink}} = \left| \frac{1}{\pi} \left\{ m \ln \frac{m}{\Lambda} - \frac{\Delta m}{2} \ln \frac{m}{\Delta m} + m + \dots \right\} \right|, \quad (6.9)$$

where $m_2 \approx m_1 = m$. We see that the kink mass stays finite in the limit $\Delta m \rightarrow 0$ as expected.

¹⁰This statement can be proven to be exact.

In fact, it is possible to obtain an exact formula for the kink mass in the limit $m_1 = m_2$. Using (5.8) it is easy to show that in the limit $\Delta m \rightarrow 0$ the kink mass is still given by Eq. (6.5). To calculate singular logarithmic terms in this expression Eq. (6.6) should be modified. Using (5.11) we get

$$\sigma_1 = -m + \Delta m \frac{2\Lambda^2}{(m + \sqrt{m^2 - 4\Lambda^2})\sqrt{m^2 - 4\Lambda^2}} + O(\Delta m^2), \quad \sigma_2^\pm = \pm 2\Lambda, \quad (6.10)$$

where now we do not assume that $m \gg \Lambda$.

Substituting this in (6.5) we get the exact result

$$M^{\text{kink}} = \left| \frac{1}{2\pi} \left\{ m \ln \frac{m + \sqrt{m^2 - 4\Lambda^2}}{m - \sqrt{m^2 - 4\Lambda^2}} + 2\sqrt{m^2 - 4\Lambda^2} \right\} \right|. \quad (6.11)$$

For comparison we quote the kink mass on the non-Abelian string in the U(2) gauge theory with $N_f = 2$ in the $r = 2$ vacuum [14],

$$\begin{aligned} M_{r=N}^{\text{kink}} &= \left| \frac{1}{2\pi} \left\{ \Delta m \ln \frac{\Delta m + \sqrt{\Delta m^2 + 4\Lambda^2}}{\Delta m - \sqrt{\Delta m^2 + 4\Lambda^2}} - 2\sqrt{\Delta m^2 + 4\Lambda^2} \right\} \right| \\ &= \left| \frac{1}{\pi} \left\{ \Delta m \ln \frac{\Delta m}{\Lambda} - \Delta m \right\} + \frac{i}{2} \Delta m + \dots \right|. \end{aligned} \quad (6.12)$$

We see that the kink masses on the strings in the two vacua are different. Both kink masses coincide with the bulk monopole masses in the corresponding vacua.

7 Deformation potential for the world sheet theory

Now let us discuss the μ -dependent deformation potential (the second term in (4.1)) in the theory on the non-Abelian string in the $r = N - 1$ vacuum. An obvious modification of the $r = N$ world-sheet potential (3.12) is

$$V_{\text{def}}(\sigma) = 4\pi \left| \mu \sqrt{\sigma^2 - \frac{4S}{\mu}} \right|. \quad (7.1)$$

This potential correctly reproduces string tensions in the vacua determined by Eq. (5.11), see (4.2). Quasiclassically these vacua are given by (5.13) and

(5.18). At two points σ_N^\pm vacuum energy is zero. This “vacuum” corresponds to the non existing N -th string. The split of this “vacuum,” which classically corresponds to $\sigma_N \approx 0$ is due to the cut which opens up on the sigma plane. Classically this cut is invisible.

As was already mentioned, the deformation potential (7.1) breaks $\mathcal{N} = (2, 2)$ supersymmetry down to $\mathcal{N} = (0, 2)$ which is further spontaneously broken by choosing a vacuum with nonvanishing energy.

8 Conclusions

In this paper we continue explorations of the $\mathcal{N} = (0, 2)$ theories emerging on non-Abelian strings supported by $\mathcal{N} = 2$ supersymmetric QCD with the gauge group $U(N)$ and N_f flavors of quarks ($N_f \geq N$). $\mathcal{N} = 2$ supersymmetry is broken down to $\mathcal{N} = 1$ by a small deformation: a small mass term for the adjoint matter. The beginning of this program was reported in Ref. [11] in which non-Abelian strings were considered in the $r = N$ vacuum. (Remember, r is the number of condensed (s)quarks in the large quark mass limit.)

In [11] we obtained a weighted CP model on the string world sheet. Non-perturbative corrections which determine the BPS spectrum could be derived “inside” this two-dimensional model *per se*.

We discover that the situation drastically changes in passing from $r = N$ to $r = N - 1$. In the $r = N - 1$ case which is our main focus, the low-energy two-dimensional theory on the string world-sheet receives nonperturbative corrections from the bulk, through the bulk gaugino condensate. Classically the world-sheet theory is still a weighted CP model. However, a nonvanishing gluino condensate in the bulk affects the target space on the world sheet, generating additional nonperturbative effects that lie “outside” the original classical model. We calculated these additional nonperturbative effects deforming the weighted CP model on the world sheet by virtue of the method of resolvents suggested by Gaiotto, Gukov and Seiberg for surface defects [1].

The 2D-4D correspondence (the coincidence of spectra of two-dimensional kinks and four-dimensional monopoles) remains valid in the BPS sector.

The target space deformation in the world-sheet model after penetration of the bulk corrections remains unidentified. We managed to derive the exact twisted two-dimensional superpotential bypassing this stage. Such an identification is a task for the future. Another obvious direction for future

work is generalization of the construction presented here to strings in the generic r vacua of μ -deformed $\mathcal{N} = 2$ QCD.

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Appendix A: Roots of the Seiberg-Witten curve in the $r = N - 1$ vacuum

To identify the $r = N - 1$ vacuum in terms of the SW curve

$$y^2 = \prod_{P=1}^N (x - \phi_P)^2 - 4 \left(\frac{\Lambda}{\sqrt{2}} \right)^{2N-N_f} \prod_{A=1}^{N_f} \left(x + \frac{m_A}{\sqrt{2}} \right) \quad (\text{A.1})$$

we have to find ϕ such that the curve factorizes as in (2.15) and at large quark masses $(N - 1)$ values of ϕ_P are determined by masses, see (2.12).

Let us calculate ϕ_P 's and the roots e_P 's in the quasiclassical approximation (5.12). First consider "large"

$$\phi_P \approx -m_P/\sqrt{2}, \quad P = 1, \dots, (N - 1).$$

Let us calculate the correction to a particular ϕ_{P_0} , $P_0 = 1, \dots, (N - 1)$. The SW curve reduces to

$$y^2 \sim (x - \phi_{P_0})^2 - 4 \left(\frac{\Lambda}{\sqrt{2}} \right)^{2N-N_f} \frac{\prod_{K=N}^{N_f} \frac{m_K - m_{P_0}}{\sqrt{2}}}{\prod_{P \neq P_0} \frac{m_P - m_{P_0}}{\sqrt{2}}} \frac{2}{m_{P_0}^2} \left(x + \frac{m_{P_0}}{\sqrt{2}} \right), \quad (\text{A.2})$$

where $P = 1, \dots, N - 1$. Now we look for ϕ_{P_0} which ensures that the corre-

sponding root e_{P_0} is a double root. We then get

$$\begin{aligned}\sqrt{2}\phi_{P_0} &= -m_{P_0} - \Lambda^{2N-N_f} \frac{\prod_{K=N}^{N_f} (m_K - m_{P_0})}{\prod_{P \neq P_0} (m_P - m_{P_0})} \frac{1}{m_{P_0}^2} + \dots, \\ \sqrt{2}e_{P_0} &= -m_{P_0} + \Lambda^{2N-N_f} \frac{\prod_{K=N}^{N_f} (m_K - m_{P_0})}{\prod_{P \neq P_0} (m_P - m_{P_0})} \frac{1}{m_{P_0}^2} + \dots.\end{aligned}\quad (\text{A.3})$$

The value of the double root $\sqrt{2}e_{P_0}$ here coincides with the VEV σ_{P_0} (see (5.13)) calculated from the world-sheet theory.

Now consider “small” unpaired roots e_N^\pm . Assuming that x is small compared to the quark masses we obtain for the SW curve

$$y^2 \sim (x - \phi_N)^2 - 4 \left(\frac{\Lambda}{\sqrt{2}} \right)^{2N-N_f} \frac{\prod_{K=N}^{N_f} \frac{m_K}{\sqrt{2}}}{\prod_{P=1}^{N-1} \frac{m_P}{\sqrt{2}}}. \quad (\text{A.4})$$

The condition (2.16) gives $\phi_N \approx 0$ and

$$\sqrt{2}e_N^\pm \approx \pm 2 \sqrt{\Lambda^{2N-N_f} \frac{\prod_{K=N}^{N_f} m_K}{\prod_{P=1}^{N-1} m_P}}. \quad (\text{A.5})$$

This result shows that the combination under the square root in (5.16) is precisely $e_N^2/2$. Then (5.19) follows.

Appendix B: Monopole mass in the $r = N - 1$ vacuum

The monopole mass is given by

$$M_{PP'}^{\text{monopole}} = \left| \frac{\sqrt{2}}{2\pi i} \oint_{\beta_{PP'}} d\lambda_{SW} \right|, \quad P, P' = 1, \dots, N, \quad (\text{B.1})$$

see (3.15). Here the SW differential is defined as [41, 42]

$$d\lambda_{SW} = \frac{x dP}{y} - \frac{xP}{2y} \frac{dQ}{Q} + \frac{x}{2} \frac{dQ}{Q} \quad (\text{B.2})$$

where P and Q are polynomials which enter the SW curve (2.10),

$$P(x) = \prod_{P=1}^N (x - \phi_P), \quad Q(x) = \prod_{A=1}^{N_f} \left(x + \frac{m_A}{\sqrt{2}} \right). \quad (\text{B.3})$$

The residues at $x = -m_A/\sqrt{2}$ in (B.2) are fixed to be integers of m_A . Moreover, the SW differential (B.2) does not have poles at $x = e_P$.

Consider the simplest case of the $U(2)$ theory with $N_f = 2$. The SW curve in the $r = N - 1 = 1$ vacuum factorizes as in (2.15). The monopole mass takes the form

$$M^{\text{monopole}} = \left| \frac{1}{\pi} \int_{\sqrt{2}e_1}^{\sqrt{2}e_2^{\pm}} z dz \left\{ \frac{1}{\sqrt{z^2 - 2e_2}} \right. \right. \\ \left. \left. - \frac{1}{2} \frac{\sqrt{m_1^2 - 2e_2^2}}{\sqrt{z^2 - 2e_2^2}(z + m_1)} + \frac{1}{2} \frac{\sqrt{m_2^2 - 2e_2^2}}{\sqrt{z^2 - 2e_2^2}(z + m_2)} \right\} \right|, \quad (\text{B.4})$$

where the integration variable $z = \sqrt{2}x$. We see that the integral representations for the four-dimensional monopole and two-dimensional kink masses are the same, see (6.2).

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